Solutions to Exam 1

1. a) $p(t) = (1,-5,2) + t(-2,0,3) = (1-2t, -5, 2+3t)$.

b) $\vec{v} = (1,2,3) - (-1,0,2) = (2,2,1)$ is a possible direction vector.
Thus, $L(t) = (-1,0,2) + t(2,2,1)$ is a parametric equation of a line that goes through the points $(1,2,3)$ and $(-1,0,2)$.
Another possibility is $p(t) = (1,2,3) + t(-2,-2,-1)$.
Note: There are infinitely many correct solutions.

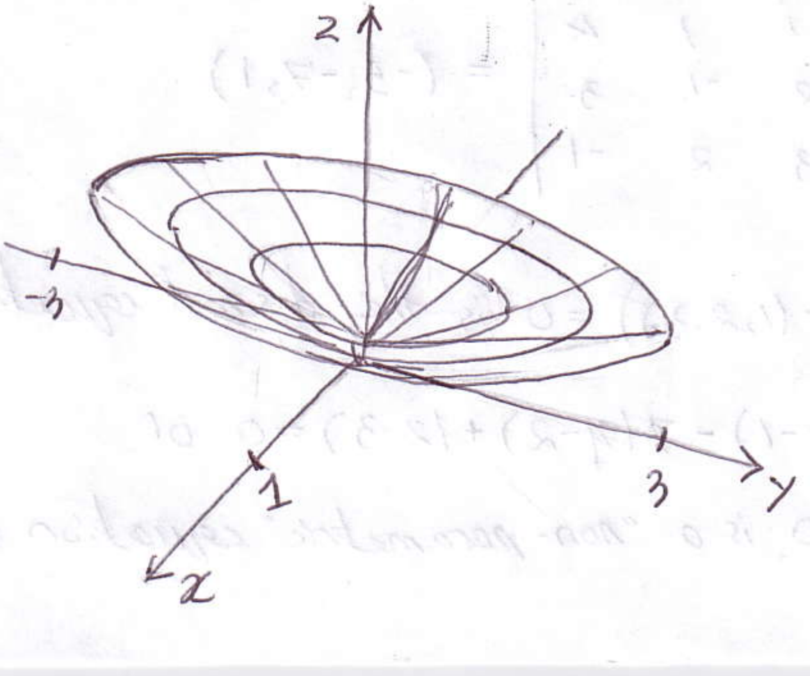
c) $\vec{v} = (2,-1,2)$
$p(t) = (1,2,3) + t(2,-1,2)$

d) $\vec{v} = (5,1,-4)$, $\vec{w} = (2,-2,2)$ Now $\vec{v}\cdot\vec{w} = 5\cdot 2 + 1(-2) + (-4)\cdot 2 = 0$

Hence the line $L(t)$ is perpendicular to $S(t)$.

e) P is a line, because $(-2,-14,4) = -2(1,7,-2)$ which means that $(1,7,-2)$ and $(-2,-14,4)$ are linearly dependent.

2. The surface is an inverted elliptical cone



3. Let $S=\{(x,y): x^2+y^2=1\}$ and suppose $T(x,y)=(6x,2y)$.

Then $T(S)=\{T(x,y): x^2+y^2=1\} = \{(6x,2y): x^2+y^2=1\} =$

$= \{(6x,2y): \left(\frac{6x}{6}\right)^2+\left(\frac{2y}{2}\right)^2=1\} = \{(u,v): \frac{u^2}{6^2}+\frac{v^2}{2^2}=1\}$.

Hence $T(S)$ is an ellipse.

The volume inside the ellipse is $6\cdot 2\cdot \pi 1^2 = 12\pi$.

4. Let $\vec{v}=(2,3)-(1,1)=(1,2)$ and $\vec{w}=(-1,0)-(1,1)=(-2,-1)$.

The area of the parallelogram that is spanned by vectors $\vec{v}$ and $\vec{w}$ is

$$\left|\det\begin{pmatrix} v \\ w\end{pmatrix}\right| = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -1+4=3$$

Since the triangle spanned by $\vec{v}$ and $\vec{w}$ has half the area of the parallelogram spanned by the same vectors, the answer is $\frac{3}{2}$.

5. a) The plane is spanned by $S(s,t)=(1,2,3)+s(2,-1,3)+$
$+t(-3,2,-1) = (1+2s-3t,\ 2-s+2t,\ 3+3s-t)$

b) Let $\vec{n} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ -3 & 2 & -1 \end{vmatrix} = (-5,-7,1)$

Then $\vec{n}\cdot((x,y,z)-(1,2,3))=0$ is the desired equation.

In particular, $-5(x-1)-7(y-2)+(z-3)=0$ or

$-5x-7y+z+16=0$ is a "non-parametric" equation of this plane.

6. a) $(T+S)(x,y,z) = (3x+2y-z,\ 6y,\ x-z) +$
$+ (2x+y+z,\ z,\ 3z) = (5x+3y,\ 6y+z,\ x+2z)$

b) $(S-2T)(x,y,z) = (-4x-3y+3z,\ -12y+z,\ -2x+5z)$.

c) $(TS)(x,y,z) = (6x+3y+2z,\ 6z,\ 2x+y-2z)$ because

$$\mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 6 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & 3 & 2 \\ 0 & 0 & 6 \\ 2 & 1 & -2 \end{pmatrix}$$ and therefore $$TS(x,y,z) \equiv \begin{pmatrix} 6 & 3 & 2 \\ 0 & 0 & 6 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

d) $$\mathcal{M}(T) = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 6 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

e) $$\mathcal{M}(S) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

f) $$\mathcal{M}(S)^T = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

7. a) $AB = \begin{pmatrix} 5 & 1 & 10 \\ 4 & 1 & 6 \end{pmatrix}$

b) BA is undefined.

8. $\mathcal{M}(T) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = A$. $A^{-1} = \frac{1}{3-2} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}^T =$

$= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ Therefore

$T^{-1}(x,y) = (x-y, -2x+3y)$

9. $\lim\limits_{\substack{(x,y)\to(0,0) \\ x=0}} \frac{2xy}{x^2+y^2} = \lim\limits_{y\to 0} \frac{0}{y^2} = 0$

$\lim\limits_{\substack{(x,y)\to(0,0) \\ y=x}} \frac{2xy}{x^2+y^2} = \lim\limits_{x\to 0} \frac{2x^2}{2x^2} = 1$

Therefore, the limit does not exist.

Another way to see that the limit above depends on the direction, from which $(0,0)$ is approached, is to express the limit in polar coordinates:

$\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^2 \sin 2\theta}{r^2} = \sin(2\theta)$ which

varies depending on the angle of approach

10. a) $\begin{vmatrix} 3a_{21} & 3a_{22} & 3a_{23} \\ a_{11} & a_{12} & a_{13} \\ b_{31}-a_{31} & b_{32}-a_{32} & b_{33}-a_{33} \end{vmatrix} =$

$= 3\left(\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \right) =$

$= 3(-1-2) = -9$

b) $P_w(v) = \frac{v \cdot w}{\|w\|^2} w = \frac{(0,6,3)\cdot(2,10,8)}{(\sqrt{2^2+10^2+8^2})^2}(2,10,8) =$

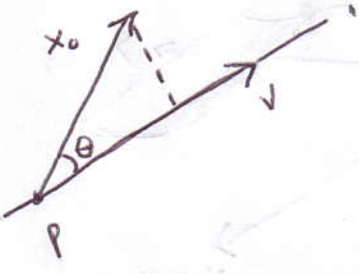
$= \frac{60+24}{2^2+10^2+8^2}(2,10,8) = \frac{15+6}{1+25+16}(2,10,8) = \frac{21}{42}(2,10,8) =$

$= (1,5,4)$.

11. $|6x+y-3z-6\cdot 1-(-2)-3\cdot(0)| = |6(x-1)+(y+2)-3(z-0)| \leq$

$\leq \|(6,1,-3)\| \, \|(x-1, y+2, z)\| < \epsilon$ Hence you should

set $\delta(\epsilon) = \frac{\epsilon}{\|(6,1,-3)\|}$.

12. 

$\frac{\|(x_0-p)\times v\|}{\|v\|} \equiv$ distance from x_0 to line $L(t)$.

$p = (1,-1,3)$ $v = (2,10,8)$ and $x_0 - p = (1,5,6) - (1,-1,3) =$
$= (0,6,3)$

Now,

$$(x_0 - p) \times v = \begin{vmatrix} i & j & k \\ 0 & 6 & 3 \\ 2 & 10 & 8 \end{vmatrix} = (18, 6, -12)$$

Hence, the distance is $\dfrac{\|(18,6,-12)\|}{\|(2,10,8)\|} = \dfrac{3\sqrt{9+1+4}}{\sqrt{1+25+16}} = \sqrt{3}$.

13. Let $u = y - 2x$, then $z = f(x,y) = e^{-(y-2x)^2} = e^{-u^2}$

z is constant whenever u is constant. In particular, if $u = c$, $z = e^{-c^2}$. Now $u = c$ is the same as $y - 2x = c$ or $y = 2x + c$.

It follows that the graph of f is a cylinder with respect to the coordinate system in which the x-axis is replaced by the line $y = 2x$.

Hence the graph of f is